Slopes of trigonal fibred surfaces and of higher dimensional fibrations

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Abstract

We give lower bounds for the slope of higher dimensional fibrations $f\colon X \longrightarrow B$ over curves under conditions of GIT-semistability of the fibres, using a generalization of a method of Cornalba and Harris. With the same method we establish a sharp lower bound for the slope of trigonal fibrations of even genus and general Maroni invariant; in particular this result proves a conjecture due to Harris and Stankova-Frenkel.

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1 Introduction and preliminaries

Given a fibration over a curve $f\colon X \longrightarrow B$ (X, B complex, projective varieties, B a smooth curve, f surjective with connected fibres) and a line bundle $\mathcal{L} = \mathcal{O}_X(L)$ on X, we can define the *slope of the pair* $(f, \mathcal{L})$ to be the quotient

$$s(f,\mathcal{L}) = \frac{L^n}{\deg f_*\mathcal{L}}$$

provided $\deg f_*\mathcal{L} \neq 0$, where n is the dimension of X. When $\mathcal{L} = \omega_f$, the relative dualizing sheaf of f, we simply call it the *slope* of f and will denote it as $s(f)$. Lower bounds for the slope have been extensively studied in the literature (e.g., [1], [4], [5], [23], [28], [14], [15]) for the case of fibred surfaces ($n = 2$) and some results are known in dimension $n = 3$ ([3], [20]).

In this paper, we study this problem using a generalized version of a theorem of Cornalba-Harris ([7], [23]). This method provides a general result to produce lower bounds of $s(f,\mathcal{L})$ provided the pair $(F, |\mathcal{L}_{|F}|)$ (where F is a general fibre of f) is semistable in the sense of Geometric Invariant Theory. In Chapter 2 we recall this result and derive the following consequences (see corollaries 2.3 and 2.5 for a more detailed statement).

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Theorem 1.1. *Under the above hypotheses, assume $Rf_*\mathcal{L}^h = 0$ for $i > 0$ and $h \gg 0$*

i) If $|\mathcal{L}_{|F}|$ induces an embedding, we have

$$L^n \geq n \frac{(L_{|F})^{n-1}}{h^0(F, \mathcal{L}_{|F})} \deg f_*\mathcal{L}.$$

ii) If $|\mathcal{L}_{|F}|$ induces an generically finite rational map onto a variety of degree d and $f_*\mathcal{L}^h$ is nef, we have

$$L^n \geq n \frac{d}{h^0(F, \mathcal{L}_{|F})} \deg f_*\mathcal{L}.$$

The method applied directly to families of canonical varieties would give very interesting higher dimensional slope inequalities. However, already in the case of surfaces it is very hard to check the stability assumption. For instance it is not known if a "general" surface of general type satisfies it or not. In the case of hypersurfaces with high enough degree, and with log-terminal singularities the stability has been proven by Tian ([26]) using methods of differential geometry. Then we can deduce a bound on the slope, when the fibres are hyperfurfaces which are *canonical*, i.e., such that its canonical map is birational (see 2.8)

Theorem 1.2. *Let $f\colon X \longrightarrow B$ be a surjective flat morphism from a $\mathbb{Q}$-factorial projective n-fold X to a smooth complete curve B. Suppose that the fibration is relatively minimal and that the general fibres F are minimal canonical varieties (of dimension $n-1$) with $p_g = n+1$, $K_F^{n-1} = n+2$, such that its canonical image has at most log-terminal singularities. Then*

$$K_f^n \geq \frac{n(n+2)}{(n+1)} \deg f_*\omega_f.$$

Examples of such fibres F are smooth hypersurfaces of $\mathbb{P}^n$ of degree $n+2$.

It is worth mentioning that this theorem is the first result proving lower bounds for the slope of fibrations of dimension higher than 3.

Eventually we give a new evidence of the necessity of the stability assumption in the C-H theorem (Remark 2.9).

Chapter 3 is devoted to the study of a particular type of fibred surfaces, the so called *trigonal* fibrations (i.e., when the general fibre is a trigonal curve). An intensively studied problem in the last decades is to find of lower bounds for the slope of fibred surfaces. In general, the so called *slope inequality* holds ([28] and [7], [23])

$$s(f) \geq 4 - \frac{4}{g}.$$

It is sharp and equality is satisfied only for certain kind of *hyperelliptic* fibrations [1], [23].

There are several reasons to conjecture that the gonality of the general fibre of f has an (increasing) influence on the lower bound of the slope (see [14], [4], [22] and Remark 3.6). So

the next natural problem in this framework is the one of studying trigonal fibrations. The main known results are the following.

(Konno, [15]) If $f\colon X \longrightarrow B$ *is a trigonal fibration of genus* $g \geq 6$*, then*

$$s(f) \geq \frac{14(g-1)}{3g+1}. \tag{1.3}$$

(Stankova-Frenkel, [22] prop.9.2 and prop.12.3) If $f\colon X \longrightarrow B$ *is a trigonal semistable fibration, then*

$$s(f) \geq \frac{24(g-1)}{5g+1}. \tag{1.4}$$

This bound is sharp, and if equality holds the general fibres have Maroni invariant ≥ 2*.*
Moreover, if g *is even and the following conditions hold:*

- *the general fibres have Maroni invariant* 0*;*
- *the singular fibres are irreducible and have only certain kind of singularities;*

then the slope satisfies the bound

$$s(f) \geq \frac{5g-6}{g}. \tag{1.5}$$

Harris and Stankova-Frenkel conjecture (Conjecture 12.1 of [22]) bound (1.5) to hold without the extra condition on singular fibres. It has to be remarked that the bounds (1.4) and (1.5), although better than (1.3), hold only for semistable fibrations i.e. for fibred surfaces such that all the fibres are semistable curves in the sense of Deligne-Mumford; this is, from the point of view of fibred surfaces, a strong restriction. Indeed, starting from any fibred surface, one can construct a semistable one by the process of semistable reduction, but the slope cannot be controlled through this process, as shown in [24].

The main result of Chapter 3 is the following (Theorem 3.3):

Theorem 1.6. *Let* $f\colon S \longrightarrow B$ *be a relatively minimal fibred surface such that the general fibre* C *is a trigonal curve of even genus* $g \geq 6$ *and the general fibre has Maroni invariant* 0*. Then the slope satisfies inequality (1.5).*

Observe that we are not assuming f to be semistable. In particular, we give a positive answer to the Harris-Stankova-Frenkel conjecture. Moreover, in Theorem 3.3, we prove at the same time that (1.5) holds for any fibration of genus 6 whose general fibres have a g^2_5, thus improving the bound proved by Konno in [15], which is 96/25.

This result can be seen as a first step when searching for an increasing dependence of the slope from the gonality of the general fibres. The assumption on the Maroni invariant assures that the fibres are general in the locus of trigonal curves, consistently with the conjectures (see Remark 3.6).

We prove this theorem applying the C-H method to a fibred 3-fold naturally associated to the fibred surface; indeed the slope of f is related to to the one of the relative quadric-hull $W \longrightarrow B$ of the trigonal fibration $f\colon S \longrightarrow B$ (cf. [15], [6]), for a suitable line bundle on it. In

the case of Maroni invariant 0, the general fibre of the hull is $\mathbb{P}^1 \times \mathbb{P}^1$, embedded as a surface of minimal degree in $\mathbb{P}^{g-1}$, this embedding being GIT semistable by a result of Kempf ([13]).

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2 The Cornalba-Harris Method and the slope of fibrations

We work over the complex field $\mathbb{C}$. Let X be a variety (an integral separated scheme of finite type over $\mathbb{C}$), with a linear system $V \subseteq H^0(X, \mathcal{L})$, for some line bundle $\mathcal{L}$ on X. Fix $h \geq 1$ and call G_h the image of the natural homomorphism

$$H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(h)) = \mathrm{Sym}^h V \xrightarrow{\varphi_h} H^0(X, \mathcal{L}^h). \tag{2.1}$$

Set $N_h = \dim G_h$ and take exterior powers $\wedge^{N_h}\mathrm{Sym}^h V \xrightarrow{\wedge^{N_h}\varphi_h} \wedge^{N_h} G_h = \det G_h$. We can see $\wedge^{N_h}\varphi_h$ as a well-defined element of $\mathbb{P}(\wedge^{N_h}\mathrm{Sym}^h V^\vee)$.

With the above notations, we call $\wedge^{N_h}\varphi_h \in \mathbb{P}(\wedge^{N_h}\mathrm{Sym}^h V^\vee)$, *the generalized h-th Hilbert point associated to the couple* (X, V). If V induces an embedding, then for $h \gg 0$ the homomorphism φ_h is surjective and it is the classical h-th Hilbert point.

Consider the standard representation $SL(s+1, \mathbb{C}) \to SL(V)$; we get an induced natural action of $SL(s+1, \mathbb{C})$ on $\mathbb{P}(\wedge^N \mathrm{Sym}^h V^\vee)$, and we can introduce the associated notion of GIT (semi)stability: we say that the h-th generalised Hilbert point of the couple (X, V) is semistable (resp. stable) if it is GIT semistable (resp. stable) with respect to the natural $SL(s+1, \mathbb{C})$-action.

We say that (X, V) is *generalised Hilbert stable (resp. semistable) if its generalised h-th Hilbert point is stable (resp. semistable) for infinitely many integers $h > 0$.*

We state a generalised version of the Cornalba-Harris theorem.

Theorem 2.2. *[[23], Theorem 1.5] Let X be a variety of dimension n and $\phi \colon X \longrightarrow B$ a flat morphism over a curve, and call F a general fibre. Let $\mathcal{L}$ be a line bundle on X. Let h be a positive integer, and assume that $(F, |\mathcal{L}_{|F}|)$ has semistable generalised h-th Hilbert point. Consider a vector subbundle $\mathcal{G}_h$ of $\phi_*\mathcal{L}^h$ such that $\mathcal{G}_h$ contains the image of the morphism of sheaves*

$$\mathrm{Sym}^h \phi_*\mathcal{L} \longrightarrow \phi_*\mathcal{L}^h,$$

and coincides with it at general $t \in B$.

Then the line bundle

$$\mathcal{F}_h := \left(\det \phi_*\mathcal{L}\ \right)^{-hN_h} \otimes (\det \mathcal{G}_h)^r,$$

where $r := h^0(F, \mathcal{L}_{|F})$, and $N_h := \mathrm{rank}\mathcal{G}_h$, is effective.

Corollary 2.3. *With the assumptions of Theorem 2.2, suppose moreover that X is pure dimensional, ϕ is proper, and that*

(1) the linear system $|\mathcal{L}_{|F}|$ induces an embedding of the general fibre F;

(2) the sheaves $R^i\phi_*\mathcal{L}^h$ vanish for $i > 0$ for h large enough[1].
Then, the following inequality holds

$$L^n \geq n \frac{(L_{|F})^{n-1}}{h^0(F, \mathcal{L}_{|F})} \deg \phi_*\mathcal{L}. \tag{2.4}$$

Proof. By the first assumption, we can apply Theorem 2.2 with $\mathcal{G}_h = \phi_*\mathcal{L}^h$. Hence, for infinitely many $h > 0$ the line bundle $\mathcal{F}_h$ is effective. Now, under our assumptions $\deg \mathcal{F}_h$ is a degree n polynomial in h with coefficients in the rational Chow ring $CH^1(X)_{\mathbb{Q}}$. Its leading coefficient has to be pseudo-effective, hence to have non-negative degree. The statement now follows from an intersection-theoretical computation. Indeed, the Riemann-Roch formula for singular varieties (cf. [8], Corollary 18.3.1) implies the following expansions:

$$N_h = \frac{(L_{|F})^{n-1}}{(n-1)!} + O(h^{n-2}),$$

and

$$\deg \phi_*\mathcal{L}^h = \deg \phi_!\mathcal{L}^h = \frac{h^n L^n}{n!} + O(h^{n-1}),$$

because the higher direct images vanish by the second assumption. □

Corollary 2.5. *With the assumptions of Theorem 2.2, suppose moreover that X is pure dimensional, ϕ is proper, and that for large enough h*

(1) the linear system $|\mathcal{L}_{|F}|$ induces a finite rational map on the image of F;

(2) the vector bundle $\phi_*\mathcal{L}^h$ is nef (i.e. every quotient has non-negative degree);

(3) the sheaves $R^i\phi_*\mathcal{L}^h$ vanish for $i > 0$.

Then the following inequality holds

$$L^n \geq n \frac{d}{h^0(F, \mathcal{L}_{|F})} \deg \phi_*\mathcal{L}, \tag{2.6}$$

where d is the degree of the image $\phi(F) \subseteq \mathbb{P}^r$.

Proof. Let h be large enough. By the nef-ness assumption on $\phi_*\mathcal{L}^h$, the degree of $\mathcal{F}_h$ is smaller or equal to the degree of $\left(\det \phi_*\mathcal{L}\ \right)^{-hN_h} \otimes \left(\det \phi_*\mathcal{L}^h\right)^r$. Then the statement follows applying Riemann-Roch for singular varieties as in the previous corollary, observing that (by the first assumption) $N_h = dh^{n-1}/(n-1)! + O(h^{n-2})$. □

In particular, using the relative canonical divisor, we can obtain the following result on the slopes of families of certain canonical varieties.

Remark 2.7. Let $\phi\colon X \longrightarrow B$ be a fibration of a normal $\mathbb{Q}$-factorial variety with at most canonical singularities over a curve. Under these assumptions K_X (and $K_\phi = K_X - \phi^*K_B$) is a Weil, $\mathbb{Q}$-Cartier divisor. We can consider its associated divisorial sheaves ω_X and ω_ϕ. Suppose that the canonical sheaf ω_X is ϕ-nef, and that on a general fibre F the canonical

[1]This happens for instance if $\mathcal{L}$ is ϕ-ample.

divisor $\omega_F = \omega_{\phi|F}$ induces a Hilbert semistable map which is finite on the image of F. Then the following inequality holds

$$K_\phi^n \geq n \, \frac{d}{p_g(F)} \deg \phi_* \omega_\phi,$$

where $p_g(F) = h^0(F, K_F)$ and d is the degree of the canonical image of the general fibre F in $\mathbb{P}^{p_g(F)-1}$. In particular, if ω_F induces a birational morphism, $d = K_F^{n-1}$.

Indeed, we can apply Corollary 2.5 to the relative canonical sheaf: $\mathcal{L} = \omega_\phi$. The second assumption is satisfied by [27], while the third one derives from the relative nefness of ω_X, using the relative version of Kawamata-Viehweg vanishing Theorem (see for instance [12], Theorem 1.2.3).

Although it is difficult to check the stability assumption for varieties of dimension bigger than 1, by a result of Tian we have

(Tian [26]) Any normal hypersurface $F \subseteq \mathbb{P}^N$ of degree $d \geq N + 2$ with only log-terminal singularities is Hilbert stable.

We will say that a variety is *canonical* if its canonical map is birational onto its image. Hence se can state

Theorem 2.8. *Let $\phi \colon X \longrightarrow B$ be a surjective flat morphism from a $\mathbb{Q}$-factorial projective n-fold X to a smooth complete curve B. Suppose that K_ϕ is ϕ-nef and that the general fibres F are minimal canonical varieties (of dimension $n-1$) with $p_g = n+1$, $K_F^{n-1} = n+2$ whose canonical image has at most log-terminal singularities. Then*

$$K_\phi^n \geq \frac{n(n+2)}{n+1} \deg \phi_* \omega_\phi.$$

Proof. It follows straightforward from the argument of Remark 2.7 and Tian's theorem. □

For instance a one-parameter family of surfaces with $p_g = 4$ $q = 0$ and $K^2 = 5$ such that the general fibre is of type (I) in Horikawa classification ([11], Theorem 1, sec.1) satisfies the conditions of the above theorem; indeed these surfaces have base-point-free birational canonical map, and their canonical image is a 5-ic surface in $\mathbb{P}^3$ with at most rational double points.

Remark 2.9. We can now give a new example that show the fact that the stability condition in the C-H method is necessary, in addition to the one given by Morrison in section 3 of [7].

In [20], Example on page 664, a fibred 3-fold $\phi \colon T \longrightarrow B$ is constructed fitting in the following diagram

$$\begin{tikzcd} T \arrow[r, "\pi"] \arrow[d, "\phi"'] & W \arrow[r, "\beta"] \arrow[dl] & \mathbb{P}_B(\phi_*\omega_\phi) \arrow[dll, "\alpha"] \\ B & & \end{tikzcd}$$

such that

- the general fibre of ϕ is a surface of general type with $p_g = 4$, $q = 0$ and $K_F^2 = 4$, and such that its canonical map is a degree 2 base point free map on to a quadric cone in $\mathbb{P}^3$.

- the map π is a smooth double cover of a $\mathbb{P}^2$-bundle W over B such that $\phi_*\omega_\phi = \alpha_*(\omega_\alpha \otimes \mathcal{L}) \oplus \alpha_*\omega_\alpha = \alpha_*(\omega_\alpha \otimes \mathcal{L})$ (because $\alpha_*\omega_\alpha = 0$, being the generic fibre of α a rational surface)

- the composition $\beta \circ \pi$ is the relative canonical map of ϕ

- the slope of ϕ is 20/7 (see [20] page 665 with e=2).

On the other hand, as $\omega_\alpha \otimes \mathcal{L}$ induces on the general fibres of α the natural map as a quadric cone in $\mathbb{P}^3$, by the same argument used by Konno in [15], Lemma 1.1, we can conclude that $R^i\alpha_*(\omega_\alpha \otimes \mathcal{L})^h = 0$ for $i, h > 0$.

It is well known that the quadric cone is Hilbert unstable. Assume nevertheless that we could apply Theorem 2.2 to ω_ϕ. Consider the morphism of sheaves

$$\mathrm{Sym}^h \phi_*\omega_\phi \longrightarrow \phi_*\omega_\phi^h.$$

We can choose as $\mathcal{G}_h$ (in the notations of Theorem 2.2) the sheaf $\alpha_*(\omega_\alpha \otimes \mathcal{L})^h$. Computing now the degree 3 coefficient of $\deg \mathcal{F}_h$, we obtain

$$\frac{K_\phi^3}{2} = \pi^*(K_\alpha + L)^3 \geq \frac{3}{2} \deg \phi_*\omega_\phi.$$

and hence the slope would be at least 3, a contradiction.

3 The slope of trigonal fibrations

Let $f : S \longrightarrow B$ be a relatively minimal fibred surface such that the general fibre C is a trigonal curve of genus g.

Remark 3.1. If C is a trigonal curve of genus g, it is a well known fact (see for instance [16] and [21]) that its canonical image lives on a Hirzebruch surface $\mathbb{F}_c = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ embedded in $\mathbb{P}^{g-1}$ as a surface of minimal degree. The surface $\mathbb{F}_c$ is the intersection of quadrics containing the canonical image of C in $\mathbb{P}^{g-1}$; from a more geometric point of view, it is the rational normal scroll generated by the lines spanned by the divisors in the g_3^1 on the canonical image of C. The number c is called the *Maroni invariant of* C; it has the same parity of g, and satisfies the following inequalities:

$$\frac{g-2}{3} \leq c \leq \frac{2g-2}{3}.$$

It has been shown in [17] that the Maroni invariant is an upper semicontinuos function on the trigonal locus $\mathcal{D}_3$, and hence a general genus g trigonal curve has Maroni invariant 0 (resp. 1) if g is even (resp. odd). The locus of points in $\mathcal{D}_3$ corresponding to curves with Maroni invariant > 1 has codimension 1 the even case, while it has strictly bigger codimension in the odd one.

We can extend the construction mentioned above on the fibres of f to a relative setting, using the so-called relative hyperquadric hull (see e.g. [15] and [6]). Consider the relative canonical image of S:

$$\begin{tikzcd} S \arrow[r, dashed, "\psi"] \arrow[d] & Y \subseteq \mathbb{P}_B(f_*\omega_f) = \mathbb{P} \arrow[dl, "\varphi"] \\ B & \end{tikzcd}$$

If $\mathcal{A} \in \mathrm{Pic}B$ is ample enough it can be easily checked that we have an epimorphism

$$H^0(\mathcal{J}_{Y,\mathbb{P}}(2) \otimes \varphi^*(\mathcal{A})) \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathcal{J}_{F,\mathbb{P}^{g-1}}(2)).$$

Let W_0 be the horizontal irreducible component of the base locus of the linear system on $\mathbb{P}$ given by the sections of $H^0(\mathcal{J}_{Y,\mathbb{P}}(2) \otimes \varphi^*(\mathcal{A}))$. Since the general fibre C is trigonal, W_0 is a threefold fibred over B by rational surfaces of minimal degree. Notice moreover that for $g \geq 5$ the singular locus of W_0 is contained in a finite number of fibres. Let W be a desingularization of W_0 and let L be the pull-back of the tautological divisor of $\mathbb{P}$ to W. We will call W the *relative quadric hull* associated to f and denote by $\phi\colon W \longrightarrow B$ the induced fibration. The fibre of ϕ over general $t \in B$ coincides with the one of W_0 (hence it is the rational normal scroll associated to the fibre of S over t).

The main facts we will need about the divisor L have been proved by Konno in [15] (cf. Lemma 1.1 and Lemma 1.2).

Proposition 3.2 (Konno)**.**
i) $\phi_*\mathcal{O}_W(L) = f_*\omega_f$*;*
ii) $R^p\phi_*\mathcal{O}_W(hL) = 0$ *for* $p, h > 0$*;*
iii) $K_f^2 \geq 2\chi_f + L^3$.

We can now state the main result of this chapter.

Theorem 3.3. *Let* $f\colon S \longrightarrow B$ *be a relatively minimal fibred surface such that the general fibre* C *is either:*

- *a trigonal curve of even genus* $g \geq 6$ *and zero Maroni invariant;*
- *a curve of genus* 6 *with a* g^2_5.

Then

$$s_f \geq \frac{5g-6}{g},$$

and this bound is sharp.

Proof. Using the relative quadric hull associated to f, the general fibre F of $\phi\colon W \longrightarrow B$ is just $\mathbb{P}^1 \times \mathbb{P}^1$ in the trigonal case and $\mathbb{P}^2$ in the case of a plane quintic. The restriction of L to F induces a complete embedding of F in $\mathbb{P}^{g-1}$ as a surface of minimal degree. This is Hilbert semistable according, for instance, to a result of Kempf (cf. [13] cor. 5.3); Moreover, $\mathcal{O}_W(L)$ has no higher relative cohomology by Proposition 3.2, (*ii*). We can therefore apply Corollary 2.3 and conclude that

$$L^3 \geq 3\frac{g-2}{g}\,\mathrm{deg}\phi_*\mathcal{O}_W(L).$$

The statement now follows using inequality (i) and (iii) of Proposition 3.2.

As for the sharpness of this bound, we refer to Example 3.4 below. $\square$

Example 3.4. Using a construction of Tan [25], we can prove that any trigonal curve with Maroni invariant *strictly* smaller than $(g+2)/9$ can be realized as the fibre of a semistable fibration over $\mathbb{P}^1$ with slope $(5g-6)/g$. Let C be a trigonal curve with Maroni invariant c. Recall that $\mathrm{Pic}\mathbb{F}_c = \mathbb{Z}[\ell] \oplus \mathbb{Z}[f]$, where ℓ is the negative section ($\ell^2 = -c$) and f is a fibre of the ruling $\mathbb{F}_c \to \mathbb{P}^1$. The class of $C \subset \mathbb{F}_c$ is $3\ell + kf$, where $k = (g+2+3c)/2$.

As proved for instance in [10]V.Cor 2.18, the linear system $\mid 3\ell + kf \mid$ is very ample if and only if $k > 6c$, that is $9c < g+2$. In this case, we can choose a general pencil in $\mid C \mid$. It has $C^2 = (3\ell + kf)^2 = 6k - 9c = 3g + 6$ base points. Let S be the blow up of $\mathbb{F}_c$ in these base points, and $f\colon S \longrightarrow \mathbb{P}^1$ the induced fibration. Computing the relative invariants, we obtain $K_f^2 = 5g - 6$ and $\chi_f = g$.

In [14], Example 4.6., other examples reaching the bound are provided, satisfying the condition that the bundle $f_*\omega_f$ is semistable.

Remark 3.5. The higher Maroni invariant cases cannot be treated with the C-H method. Recall that the general fibre of W is an Hirzebruch surface $\mathbb{F}_c$ embedded in $\mathbb{P}^{g-1}$ by the divisor $D = \ell + \frac{g+c-2}{2} f$; D is a "good" divisor in the notation of [18], and by Theorem 6.5 of the same paper, the associated embedding is Chow unstable (hence Hilbert unstable) if and only if $c > 0$.

On the other hand, Xiao's method has been applied to this setting (regardless to the Maroni invariant) by Konno in [15], and leads to the bound (1.3).

This seems to suggest that the two methods of Cornalba-Harris and of Xiao, while being surprisingly similar in the case of fibred surfaces (cf [4]), become substantially different when applied to fibrations whose total space has dimension ≥ 3.

Remark 3.6. As it is well known, gonality provides a stratification of the moduli space of smooth curves $\mathcal{M}_g$. Indeed, the loci

$$\mathcal{D}_k := \overline{\left\{[C] \in \mathcal{M}_g \text{ such that } C \text{ has a } g^1_k\right\}} \subseteq \mathcal{M}_g$$

are closed subsets of $\mathcal{M}_g$ of decreasing codimension as k goes from 2 to $[(g+3)/2]$. The curves with maximal gonality $[(g+3)/2]$ form an open set. It has been proved ([1] and [9] in the semistable case) that if $f\colon S \longrightarrow B$ is a fibred surface of odd genus and such that the general fibres have maximal gonality, then $s(f) \geq 6(g-1)/(g+1)$. Moreover, the slope inequality $s(f) \geq 4(g-1)/g$, that holds for any fibres surface, is an equality only for some hyperelliptic fibrations. It seems therefore natural to conjecture an increasing bound for the slope of fibred surfaces depending on the gonality of the general fibres. A natural guess would be that the slope of non-hyperelliptic fibred surfaces should satisfy at least the bounds for trigonal fibrations (1.3). This is however false, as observed for instance in [2] and [22]: the easiest counterexamples are provided by bielliptic surfaces of arbitrarily large genus, with slope 4.

The right question to ask when looking for a bound increasing with gonality seems to be that the fibre are "general" in the k-gonal locus: (see in particular Conjecture 13.3 of [22]). From this point of view, the bound (1.5) could be the first step of the desired sequence.

It is worth mentioning that Konno proves the same bound (1.5) in [14] (corollary 4.4) under the assumptions that the fibration is non-hyperelliptic, and that $f_*\omega_f$ *is a semistable vector bundle.* The assumption of semistability for $f_*\omega_f$ is difficult to interpret; it would be very interesting to understand whether it is connected with some kind of "genericity" of the general fibre.

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